NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3021

A METHOD OF DERIVING FREQUENCY-RESPONSE DATA FOR MOTION

OF THE CENTER OF GRAVITY FROM DATA MEASURED ON

AN AIRCRAFT AT LOCATIONS OTHER THAN

THE CENTER OF GRAVITY

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SUMMARY

A method is presented for deriving time-response and frequency-
response data for angle of attack and normal acceleration at the center
of gravity of an aircraft when these data are measured at locations on
the aircraft other than the center of gravity and when the pitching
velocity is not measured. The method involves the calculation of transfer
functions for the measured quantities and operation on these transfer
functions to derive the transfer function of the desired quantities at
the center of gravity. Time-response data at the center of gravity are
then obtained by applying a control input to the corrected transfer func-
tion. Basic aerodynamic relationships and the forms usually assumed for
the transfer functions are used in the derivation. A numerical example
is included to illustrate the application of the method.

The method appears to have particular application to missiles and
models of aircraft where the number of telemetering channels available
may limit the number of quantities that can be recorded or where, because
of space limitations, instruments cannot be placed in the most desirable
locations.

INTRODUCTION

In the determination of the dynamic stability of aircraft, it is
often desirable to flight test the aircraft or scale models of the air-
craft in order to determine transfer functions or frequency responses.
In the study of longitudinal stability, for example, the transfer func-
tions relating angle of attack, pitching velocity, and normal acceleration
to the control deflection may be desired. Frequently, however, facilities
for recording the data necessary for a general stability analysis are
limited. Guided missiles and rocket-propelled scale models that require
telemetering facilities often present such a problem. Furthermore,
distribution of such items as fuel and propulsion units in either models

or full-scale aircraft may preclude the possibility of locating recording instruments in the most desirable positions. For these reasons, test conditions under which a limited number of pertinent quantities are measured on the aircraft at positions other than those suitable for stability studies are frequently experienced.

When aircraft motion is confined to two degrees of freedom, any two measurements (such as two accelerometers; one in the nose and one in the tail) will suffice. It is customary, however, to define the short-period longitudinal motion of an aircraft with angle of attack and normal acceleration measured at the center of gravity. When any two independent quantities are measured, time histories of the quantities desired for defining the motion may theoretically be calculated from the measured transient response data. The procedure, in general, requires differentiating or integrating one or more of the recorded quantities and applying corrections point by point to the measured curves. This process is laborious, however, and may introduce inaccuracies not present in the original data. As an alternative, an approach which appears shorter and more accurate would be to calculate transfer functions for the measured quantities and to operate on these transfer functions to derive the transfer functions and transient time responses for the desired quantities. This approach is used in the method described in the present paper.

The derivation of the relationships between angle of attack and normal acceleration at the center of gravity and these quantities measured ahead of (or rearward of) the center of gravity is presented for the case in which pitching velocity is not known. (The method is greatly simplified when pitching velocity is known and consideration of this condition is included.) Pitching velocity is calculated from angle-of-attack and normal-acceleration transfer functions related to the center of gravity.

An application of this method is illustrated by use of data obtained from drop tests of a freely falling body.

SYMBOLS

$\bar{c}$ mean aerodynamic chord

C_L lift coefficient

C_{L_α} variation of lift coefficient with angle of attack, $\dfrac{\partial C_L}{\partial \alpha}$

$C_{L_{D\alpha}}$ variation of lift coefficient with rate of change of angle of attack, $\dfrac{\partial C_L}{\partial D\alpha}$, sec

$C_{L_{D\theta}}$ variation of lift coefficient with pitching velocity, $\dfrac{\partial C_L}{\partial D\theta}$, sec

C_{L_δ} variation of lift coefficient with elevator deflection, $\dfrac{\partial C_L}{\partial \delta}$

C_m pitching-moment coefficient

C_{m_α} variation of pitching-moment coefficient with angle of attack, $\dfrac{\partial C_m}{\partial \alpha}$

$C_{m_{D\alpha}}$ variation of pitching-moment coefficient with rate of change of angle of attack, $\dfrac{\partial C_m}{\partial D\alpha}$, sec

$C_{m_{D\theta}}$ variation of pitching-moment coefficient with pitching velocity, $\dfrac{\partial C_m}{\partial D\theta}$, sec

C_{m_δ} variation of pitching moment with elevator deflection, $\dfrac{\partial C_m}{\partial \delta}$

D differential operator, $\dfrac{d}{dt}$

g acceleration due to gravity, ft/sec^2

$j = \sqrt{-1}$

K_Y nondimensional radius of gyration of aircraft about axis of pitch

l_1 distance between center of gravity and angle-of-attack vane (positive forward), ft

l_2 distance between center of gravity and normal accelerometer (positive forward), ft

m numerical integer

n numerical integer or normal acceleration, g units

t time, sec

V true airspeed, ft/sec

α angle of attack, radians

γ flight-path angle, radians

δ elevator deflection, radians

θ angle of pitch, radians

μ relative-density factor, $\dfrac{\text{Mass}}{\text{Air density} \times \text{Wing area} \times \bar{c}}$

ϕ phase angle, deg

ω angular frequency, radians/sec

[ ]   performance operator notation

Subscripts:

cg   center of gravity

p   location of normal-acceleration pickup

v   location of angle-of-attack vane

Transfer-function coefficients:

$$A = 2\mu \frac{\bar{c}^2}{V^2} K_Y^2 \left(2\mu \frac{\bar{c}}{V} + C_{L_{D\alpha}}\right)$$

$$B = 2\mu \frac{\bar{c}}{V} \left(\frac{\bar{c}}{V} K_Y^2 C_{L_\alpha} - C_{m_{D\theta}} - C_{m_{D\alpha}}\right) + C_{L_{D\theta}} C_{m_{D\alpha}} - C_{L_{D\alpha}} C_{m_{D\theta}}$$

$$C = -2\mu \frac{\bar{c}}{V} C_{m_\alpha} + C_{m_\alpha} C_{L_{D\theta}} - C_{L_\alpha} C_{m_{D\theta}}$$

$$G = -2\mu \frac{\bar{c}^2}{V^2} K_Y^2 C_{L_\delta}$$

$$H = C_{m_\delta}\left(2\mu \frac{\bar{c}}{V} - C_{L_{D\theta}}\right) + C_{L_\delta} C_{m_{D\theta}}$$

$$J = C_{m_\delta}\left(2\mu \frac{\bar{c}}{V} + C_{L_{D\alpha}}\right) - C_{L_\delta} C_{m_{D\alpha}}$$

$$K = C_{L_\alpha} C_{m_\delta} - C_{m_\alpha} C_{L_\delta}$$

$$M = 2\mu \frac{\bar{c}^2}{gV} K_Y{}^2 C_{L_\delta}$$

$$N = \frac{V}{g} C_{m_\delta}\left(C_{L_{D\alpha}} + C_{L_{D\theta}}\right) - \frac{V}{g} C_{L_\delta}\left(C_{m_{D\alpha}} + C_{m_{D\theta}}\right)$$

$$P = \frac{V}{g}\left(C_{L_\alpha} C_{m_\delta} - C_{m_\alpha} C_{L_\delta}\right)$$

$\left.\begin{array}{l} E \\ F \\ X \\ Y \\ Z \end{array}\right\}$ coefficients of transfer functions obtained at locations other than the center of gravity and corresponding, respectively, to coefficients G, H, M, N, and P of transfer functions at the center of gravity.

ANALYSIS

Consider an aircraft having an angle-of-attack vane located a distance l_1 ahead of the center of gravity and a normal-acceleration sensing device located a distance l_2 ahead of the center of gravity. The pitching velocity is not recorded. It is assumed that the aircraft, when disturbed from trim, will remain within the linear range of the pertinent stability derivatives, that the perturbation angles are small, and that the structure is rigid. The sign convention which relates the quantities used herein is shown in figure 1. From this figure the following basic longitudinal relationship may be seen to exist:

$$gn \approx V\, D\gamma = V(D\theta - D\alpha) \tag{1}$$

The relationship between the normal acceleration at the center of gravity and the point at which the normal acceleration is recorded may be written as follows:

$$n_{cg} = n_p - \frac{l_2}{g} D^2\theta \tag{2}$$

A similar relationship may be obtained by relating the angle of attack at the center of gravity to the angle of attack measured at the angle-of-attack vane

$$\alpha_{cg} = \alpha_v + \frac{l_1}{V} D\theta \tag{3}$$

In cases in which the pitching velocity is known, equations (2) and (3) in transfer-function form may be solved to obtain the transfer functions of angle of attack and normal acceleration at the center of gravity. If the pitching velocity is not known, further transformation of the equations is required. Multiplying equation (3) by VD/g gives

$$\frac{V}{g}(D\alpha)_{cg} = \frac{V}{g}(D\alpha)_v + \frac{l_1}{g}D^2\theta \tag{4}$$

and combining equations (1) and (4) to eliminate $(D\alpha)_{cg}$ gives

$$n_{cg} + \frac{V}{g}(D\alpha)_v = \left(\frac{V}{Dg} - \frac{l_1}{g}\right)D^2\theta \tag{5}$$

The combination of equations (2) and (5) to eliminate $D^2\theta$ gives

$$n_{cg} = \frac{1}{(l_1 - l_2)D - V}\left\{\left(l_1D - V\right)n_p + \left(l_2\frac{V}{g}D^2\right)\alpha_v\right\} \tag{6}$$

which is the relationship between the two known quantities of angle of attack and normal acceleration located at points other than the center of gravity and the unknown quantity of normal acceleration at the center of gravity.

For the relationship of angle of attack at the center of gravity in terms of known quantities, equations (2) and (3) may be combined to eliminate θ

$$\alpha_{cg} = \alpha_v - \frac{l_1g}{l_2VD}\left(n_{cg} - n_p\right) \tag{7}$$

where α_v and n_p are determined from flight records and n_{cg} is given by equation (6).

Rearranging the relationships of equations (6) and (7) gives a new set of relationships between the known and unknown quantities as

$$\alpha_{cg} = \frac{1}{V - (l_1 - l_2)D}\left\{\left(\frac{l_1g}{V}\right)n_p + \left(l_2D + V\right)\alpha_v\right\} \tag{8}$$

$$n_{cg} = n_p - \frac{l_2VD}{l_1g}\left(\alpha_{cg} - \alpha_v\right) \tag{9}$$

Equations (6) and (7), or (6) and (8), or (8) and (9) may be used in the determination of angle of attack and normal acceleration at the center of gravity inasmuch as these relationships are interchangeable.

Transfer-Function Concept

In the application of this method, equations (6), (7), (8), and (9) are used in transfer-function form. The reader is assumed to be familiar with the relationship between the time and frequency realms and with the transfer-function concept of expressing the dynamic characteristics of a system. Reference 1 presents the theory of this concept and references 2 and 3 describe several methods of obtaining transfer functions from flight data. Equations (6) and (7) written in terms of the transfer functions relating normal acceleration and angle of attack to elevator deflection become, respectively,

$$
\left[\frac{n}{\delta}\right]_{cg} = \frac{1}{(l_1 - l_2)D - V}\left\{(l_1 D - V)\left[\frac{n}{\delta}\right]_p + \frac{l_2 VD^2}{g}\left[\frac{\alpha}{\delta}\right]_v\right\} \tag{10}
$$

$$
\left[\frac{\alpha}{\delta}\right]_{cg} = \left[\frac{\alpha}{\delta}\right]_v - \frac{l_1 g}{l_2 VD}\left(\left[\frac{n}{\delta}\right]_{cg} - \left[\frac{n}{\delta}\right]_p\right) \tag{11}
$$

where brackets denote a performance operator.

It should be pointed out that no assumption need be made as to the form of the transfer functions. Frequency response at the center of gravity of the airplane may be obtained from the frequency response measured at locations other than the center of gravity by the use of equations (10) and (11) (or equations (8) and (9) written in operator notation) when the substitution of $j\omega$ for D is made in the equations. However, this approach will not be expanded in this paper.

In order to illustrate the transfer-function approach, equations (10) and (11) are solved by using transfer functions as obtained by stability theory when the motions of the aircraft are confined to two degrees of freedom. This assumption is made when only the short-period longitudinal mode is of interest. If it is desired to assume transfer-function forms of higher order than are assumed herein, the method should still be applicable.

The equations of motion for two degrees of freedom (constant airspeed and no structural modes of oscillation) are expressed as follows:

$$2\mu \frac{\bar{c}^2}{V^2} K_Y{}^2 D^2\theta - C_{m_{D\theta}}D\theta - C_{m_\alpha}\alpha - C_{m_{D\alpha}}D\alpha = C_{m_\delta}\delta \qquad (12)$$

$$2\mu \frac{\bar{c}}{V}(D\theta - D\alpha) - C_{L_{D\theta}}D\theta - C_{L_\alpha}\alpha - C_{L_{D\alpha}}D\alpha = C_{L_\delta}\delta \qquad (13)$$

Through use of determinants, each of the two dependent variables α and $D\theta$ may be related to the independent variable δ in the following form:

$$\frac{\alpha}{\delta} = \frac{GD + H}{AD^2 + BD + C} \qquad (14)$$

$$\frac{D\theta}{\delta} = \frac{JD + K}{AD^2 + BD + C} \qquad (15)$$

and by the use of equation (1), the relationship between n and δ may be shown to be of the form

$$\frac{n}{\delta} = \frac{MD^2 + ND + P}{AD^2 + BD + C} \qquad (16)$$

where A, B, C, G, H, and the other transfer-function coefficients are composed of combinations of the parameters of equations (12) and (13) and are defined in terms of these parameters in the list of symbols. It may be shown that the denominator of the longitudinal transfer functions (the characteristic equation) expresses the damping and frequency of the free short-period longitudinal motion of the aircraft and is common to all such transfer functions regardless of the location on the aircraft at which the variable under consideration is measured or the input and response that are related.

In the application of this method, it is assumed that transfer functions relating elevator deflection to the response of angle of attack and normal acceleration at points other than the center of gravity have been obtained by one of the several means available. The transfer functions discussed herein are symbolized by the following equations:

$$\left[\frac{\alpha}{\delta}\right]_v = \frac{ED + F}{AD^2 + BD + C}$$

$$\left[\frac{\alpha}{\delta}\right]_{cg} = \frac{GD + H}{AD^2 + BD + C}$$

$$\left[\frac{n}{\delta}\right]_p = \frac{XD^2 + YD + Z}{AD^2 + BD + C}$$

$$\left[\frac{n}{\delta}\right]_{cg} = \frac{MD^2 + ND + P}{AD^2 + BD + C}$$

$$\left[\frac{D^2\theta}{\delta}\right] = \frac{JD^2 + KD}{AD^2 + BD + C}$$

(17)

Reference to equations (2) and (3) shows that the transfer functions at the points of measurement and the corresponding transfer functions at the center of gravity should be of the same form.

Transfer Functions at the Center of Gravity

When the appropriate forms of the transfer functions are used, equations (10) and (11) are expressed in terms of the known and unknown transfer coefficients. Equation (10), written in terms of equation (17), becomes

$$\left[\frac{MD^2 + ND + P}{AD^2 + BD + C}\right] = \frac{1}{(l_1 - l_2)D - V}\left\{(l_1 D - V)\left[\frac{XD^2 + YD + Z}{AD^2 + BD + C}\right] + \frac{l_2 V D^2}{g}\left[\frac{ED + F}{AD^2 + BD + C}\right]\right\}$$

(18)

Equation (18) indicates that, with the approach used herein, a quadratic numerator and denominator on the left-hand side of the equation have been set equal to a cubic numerator and denominator on the right-hand side of the equation. Obviously, there must exist a linear factor common to both the numerator and denominator of the right-hand side of equation (18) which can be canceled to leave an expression with only quadratic factors.

Multiplying both sides of equation (18) by the factor $\left(l_1 - l_2\right)D - V$ and equating coefficients of equal powers of the operator D yields four relationships

$$- VP = - VZ \tag{19}$$

$$\left(l_1 - l_2\right)P - VN = - VY + l_1 Z \tag{20}$$

$$\left(l_1 - l_2\right)N - VM = - VX + l_1 Y + \frac{l_2 V}{g} F \tag{21}$$

$$\left(l_1 - l_2\right)M = l_1 X + \frac{l_2 V}{g} E \tag{22}$$

A second set of relationships may be obtained by following a similar process with equation (11). Substitution of the transfer-function forms of equation (17) into equation (11) gives

$$\left[\frac{GD + H}{AD^2 + BD + C}\right] = \left[\frac{ED + F}{AD^2 + BD + C}\right] - \frac{l_1 g}{l_2 VD}\left[\frac{\left(MD^2 + ND + P\right) - \left(XD^2 + YD + Z\right)}{AD^2 + BD + C}\right] \tag{23}$$

and equating like coefficients gives the following three relationships:

$$0 = - \frac{l_1 g}{l_2 V}(P - Z) \tag{24}$$

$$H = F - \frac{l_1 g}{l_2 V}(N - Y) \tag{25}$$

$$G = E - \frac{l_1 g}{l_2 V}(M - X) \tag{26}$$

It is immediately obvious that equations (19) and (24) are equivalent to a single equation $P = Z$. Furthermore, examination of equation (22) shows that the right-hand side of equation (26) may be replaced by the expression $- \frac{g}{V} M$; therefore

$$G = - \frac{g}{V} M \tag{27}$$

This relationship can also be seen by referring to the expressions for G and M in the section entitled "Symbols." Hence, equations (22) and (26) are parametric representations of equation (27) and may be replaced by this single equation. For purposes of calculation, the equations which should be used are equations (19), (20), (21), (25), and (27). Although these equations form the basic linearly independent set of equations, equations (22) and (26) would also be satisfied if the transfer functions $\left[\dfrac{n}{\delta}\right]_p$ and $\left[\dfrac{\alpha}{\delta}\right]_v$ were known exactly. However, in the practical application where flight data are used, inaccuracies are inherent in the determination of the measured coefficients. In the procedure used to determine coefficients relating to motion of the center of gravity, the inherent inaccuracies should be minimized.

Both equation (22) and equation (26) contain the coefficient E which is the coefficient of the first power of the operator D in the angle-of-attack transfer function measured at the vane. Experience has shown that E and the associated coefficient of the transfer function at the center of gravity G are usually so small that they have little effect on airplane response and are therefore usually obtained with low percentage accuracy. Hence, although equations (22) and (26) should be numerically compatible with the other equations (equations (19) to (27)), this numerical check may be poor because of the inherent inaccuracies in the measurement of the coefficient E.

The four relationships, equations (19), (20), (21), and (25), may be rearranged slightly to express each unknown coefficient as a function of the known coefficients. These expressions are

$$H = F + \frac{l_1 g}{V^2}\, Z \tag{28}$$

$$P = Z \tag{29}$$

$$N = Y - \frac{l_2}{V}\, Z \tag{30}$$

$$M = X - \frac{l_2}{g}\, F - \frac{l_2}{V}\, Y - \frac{l_2}{V^2}\left(l_1 - l_2\right)Z \tag{31}$$

The five equations, equations (27) to (31), express the coefficients of the numerators of the angle-of-attack and normal-acceleration transfer functions related to the center of gravity in terms of measured transfer-function coefficients, the distance between their point of measurement and the center of gravity, and the airspeed at which they were measured.

The inverse Laplace transformation may be applied to the determined transfer functions in order to obtain time-response data at the center of gravity in response to control inputs that have Laplace transformations.

Pitching-Velocity Transfer Function

The transfer function relating pitching velocity to elevator deflection may be obtained in a manner similar to that described for obtaining transfer functions involving normal acceleration and angle of attack. Starting with any one of the three basic equations involving pitching velocity (equations (1), (2), and (3)), the method of equating transfer-function coefficients of like powers of the operator D is applied. In order to express the coefficients J and K in terms of the measured coefficients only, equations (27) to (31) derived in the preceding section are then substituted for the derived coefficients to give

$$J = F + \frac{g}{V} Y + \frac{g}{V^2}\left(l_1 - l_2\right)Z \qquad (32)$$

$$K = \frac{g}{V} Z \qquad (33)$$

The equations used to determine the unknown coefficients (equations (28) to (33)) may be seen to express each unknown coefficient as a function of the known coefficient of the same power of D, plus a correction term. Other relationships between the coefficients may be readily obtained by combining in various ways the equations already presented. The effects of using other groups of equations on the accuracy of the numerical results are mentioned in the following section.

EXAMPLE OF APPLICATION

As an illustration of the method presented herein, use is made of data obtained from drop tests of a freely falling model for which the response in angle of attack to a step elevator input was measured 5.51 feet ahead of the center of gravity $\left(l_1\right)$ and for which the response in normal acceleration was measured 2.165 feet ahead of the center of gravity $\left(l_2\right)$. The pitching velocity was not recorded. The actual flight data were recorded at a constantly decreasing altitude and increasing Mach number. These conditions were reflected in the time responses as varying frequency, varying damping, and a slight trim change. The response data, however, were adjusted for the condition of constant altitude, constant Mach number, and zero trim change and were approximated with the following analytical expressions for a true velocity of 885 feet per second:

$\alpha(t) = 0.01745 - e^{-1.16t}(-0.00486 \sin 9.932t + 0.01745 \cos 9.932t)$, radians

$n(t) = 0.238 - e^{-1.16t}(0.356 \sin 9.932t + 0.17647 \cos 9.932t)$, g units

$\delta(t) = -0.00902$, radians

A plot of these analytical expressions is shown in figure 2.

Making the Laplace transformation (ref. 1) of these equations and dividing the transformations of the first two equations by the transformation of the third equation gives transfer functions relating angle of attack and normal acceleration to elevator deflection at the points of measurement

$$\left[\frac{\alpha}{\delta}\right]_V = \frac{3.109D - 193.40}{D^2 + 2.32D + 99.99}, \frac{\text{radian}}{\text{radian}}$$

$$\left[\frac{n}{\delta}\right]_P = \frac{-6.819D^2 + 0.7266D - 2637.8}{D^2 + 2.32D + 99.99}, \frac{\text{g units}}{\text{radian}}$$

These transfer functions are in the normalized form in which all the coefficients have been divided by the inertia coefficient A. The substitution of $D = j\omega$ in these transfer functions produces frequency-response relationships that are plotted in figure 3 as amplitude ratio and phase angle. The NACA sign convention used in this paper (fig. 1) defines a positive elevator deflection as one for elevator trailing edge down, and therefore, a positive elevator deflection will, in general, produce negative static responses. In order to conform with the usual practice of plotting frequency-response curves, the phase angles have been shifted $180°$ (zero phase angle at zero frequency). The unusual trend in the phase angles shown in figure 3(a) is caused by the negative coefficient of the D^2 term in the numerator of the transfer function relating normal acceleration to elevator deflection.

The application of equations (27) to (31) to the coefficients of the transfer functions obtained ahead of the center of gravity gives the following coefficients corrected to the center of gravity:

$$P/A = -2637.8$$

$$N/A = 7.179$$

$$M/A = 6.207$$

$$G/A = -0.226$$

$$H/A = -194.00$$

In order to illustrate the effect on the coefficients M and G when equations involving E are used, calculations of the coefficients were made with equations (22) and (26). Equation (22) gave

$$M/A = 44.07$$

Equation (26) gave

$$G/A = -1.6 \qquad\qquad (M/A = 44.07)$$

$$G/A = 1.9 \qquad\qquad (M/A = 6.207)$$

These values of the coefficients M and G differ considerably from those found by use of equations (27) and (31) and give illogical values of amplitude ratio at high frequencies in that a much greater value of control effectiveness would be required than could be obtained from the freely falling model. With reference to the stability parameters that define G/A, a positive sign for this coefficient is likewise illogical. A check on logical values of the high frequency coefficients M/A and G/A was made by noting that from stability theory (see transfer-function coefficients in the section on symbols)

$$\frac{M}{A} \approx \frac{V^2}{2\mu g\bar{c}} C_{L_\delta} \qquad\qquad \frac{G}{A} \approx \frac{-V}{2\mu\bar{c}} C_{L_\delta}$$

By the use of these relationships, values of the parameter C_{L_δ} (based on wing area) were found to be in good agreement with theory when equations not involving E (i.e., equations (27) and (31)) were used for obtaining values of G/A and M/A.

Additional numerical solutions for the coefficients of the transfer functions were obtained by the use of several other sets of equations which may be derived from combinations of equations (27) to (33). All of these sets of equations appeared to be less well conditioned for obtaining accurate results than the equations presented because these sets of equations frequently expressed the values of unknown coefficients in terms of known coefficients of other powers of D to which they are not directly related physically. In some cases these other combinations of equations yielded numerical answers for the unknown coefficients that differed from logical values by as much as a factor of 1000.

The coefficients of the numerator of the pitching-velocity transfer function may be obtained by using equations (32) and (33) to give

$$J/A = -193.74$$

$$K/A = -95.97$$

The complete transfer functions can now be given by the following equations:

$$\left[\frac{\alpha}{\delta}\right]_{cg} = \frac{-0.226D - 194.00}{D^2 + 2.32D + 99.99}, \frac{radian}{radian}$$

$$\left[\frac{n}{\delta}\right]_{cg} = \frac{6.207D^2 + 7.179D - 2637.8}{D^2 + 2.32D + 99.99}, \frac{g \; unit}{radian}$$

$$\left[\frac{D\theta}{\delta}\right] = \frac{-193.74D - 95.97}{D^2 + 2.32D + 99.99}, \frac{radian/second}{radian}$$

The frequency response of the model described by these transfer functions is shown in figure 4.

It may be noted that the significant changes in the transfer functions resulting from the transformation to the center of gravity occurred in the coefficients of the highest order terms (of the numerators). The negative sign of the coefficient G and the positive sign of the coefficient M obtained in the transformation are in agreement with stability theory. The phase-angle variation of angle of attack related to elevator deflection is such that in the frequency range shown (fig. 4 (b)), the lag appears to approach a constant value of 180°. At higher frequencies, however, where the high frequency coefficient becomes dominant, the phase-angle variation approaches a constant value of 90° lag. It may also be noted that at the natural frequency of the freely falling model, the phase-angle variation of the ratio of pitching velocity to elevator deflection is about zero; whereas normal-acceleration and angle of attack are lagging elevator deflection by about 90°.

The foregoing results may be further illustrated in the time realm by the application of the inverse Laplace transformation to the derived transfer functions. For an elevator step input of the same magnitude as that used in the test, δ(t) = -0.00902 radian, the analytical expressions as functions of time will be

$$\alpha(t) = 0.0175 - 0.0176 e^{-1.16t} \sin(9.932t - 96.6°), \text{ degrees}$$

$$n(t) = 0.2380 - 0.30592 e^{-1.16t} \sin(9.932t + 84.605°), \text{ g units}$$

$$D\theta(t) = 0.00866 + 0.1752 e^{-1.16t} \sin(9.932t - 2.83°), \text{ radians/second}$$

These time responses together with the time responses from figure 2 are shown in figure 5.

Inspection of figure 5 indicates that the angle of attack was affected very little by the transformation; whereas normal acceleration was changed considerably. The initial positive jump in normal acceleration at $t = 0$ experienced by the freely falling model at a point 2.165 feet ahead of the center of gravity as compared with the negative jump at $t = 0$ experienced at the center of gravity is due to the effectively instantaneous change in tail loading as a result of the step deflection. The angular acceleration produced by application of the tail load resulted in a component of linear acceleration at the point of measurement which was larger in magnitude than and opposite in direction to the actual linear acceleration produced by the tail load, which is indicated by the normal acceleration calculated at the center of gravity.

CONCLUDING REMARKS

A method is presented for analyzing longitudinal response data when pitching velocity is not known and normal acceleration and angle of attack are measured at locations other than the center of gravity. By means of this method frequency-response data or transfer functions for angle of attack and normal acceleration are obtained. From these transfer functions the entire time response may be obtained for these quantities at the center of gravity. This method also provides a means of obtaining pitching velocity when normal acceleration and angle of attack are not recorded at the same location in the airplane. Although the method as applied used transfer functions obtained from stability theory for a system with two degrees of freedom, the analysis was general and the same principal should be applicable to systems of higher order or of a greater number of degrees of freedom.

The numerical example presented indicates that care must be used in selecting a well conditioned set of equations for the coefficients of the transfer function if accurate results are to be obtained.

This method appears to have particular application to missiles and models of aircraft where the number of telemetering channels available

P

may limit the number of quantities that can be recorded or where, because
of space limitations, instruments cannot be placed in the most desirable
locations.

Langley Aeronautical Laboratory,
     National Advisory Committee for Aeronautics,
          Langley Field, Va., July 29, 1953.

<div align="center">REFERENCES</div>

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     Studied by the Laplace Transformation. Lumped-Constant Systems.
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2. Greenberg, Harry: A Survey of Methods for Determining Stability
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3. Donegan, James J., and Pearson, Henry A.: Matrix Method of Determining
     the Longitudinal-Stability Coefficients and Frequency Response of an
     Aircraft From Transient Flight Data. NACA Rep. 1070, 1952. (Super-
     sedes NACA TN 2370.)
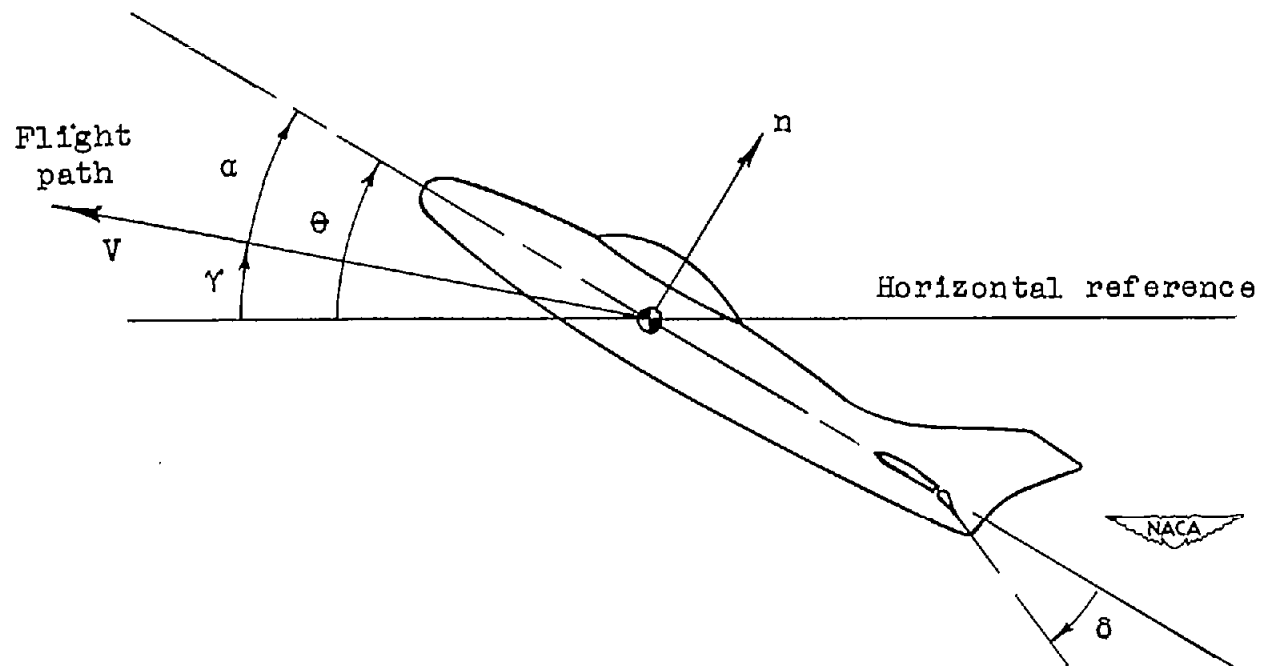
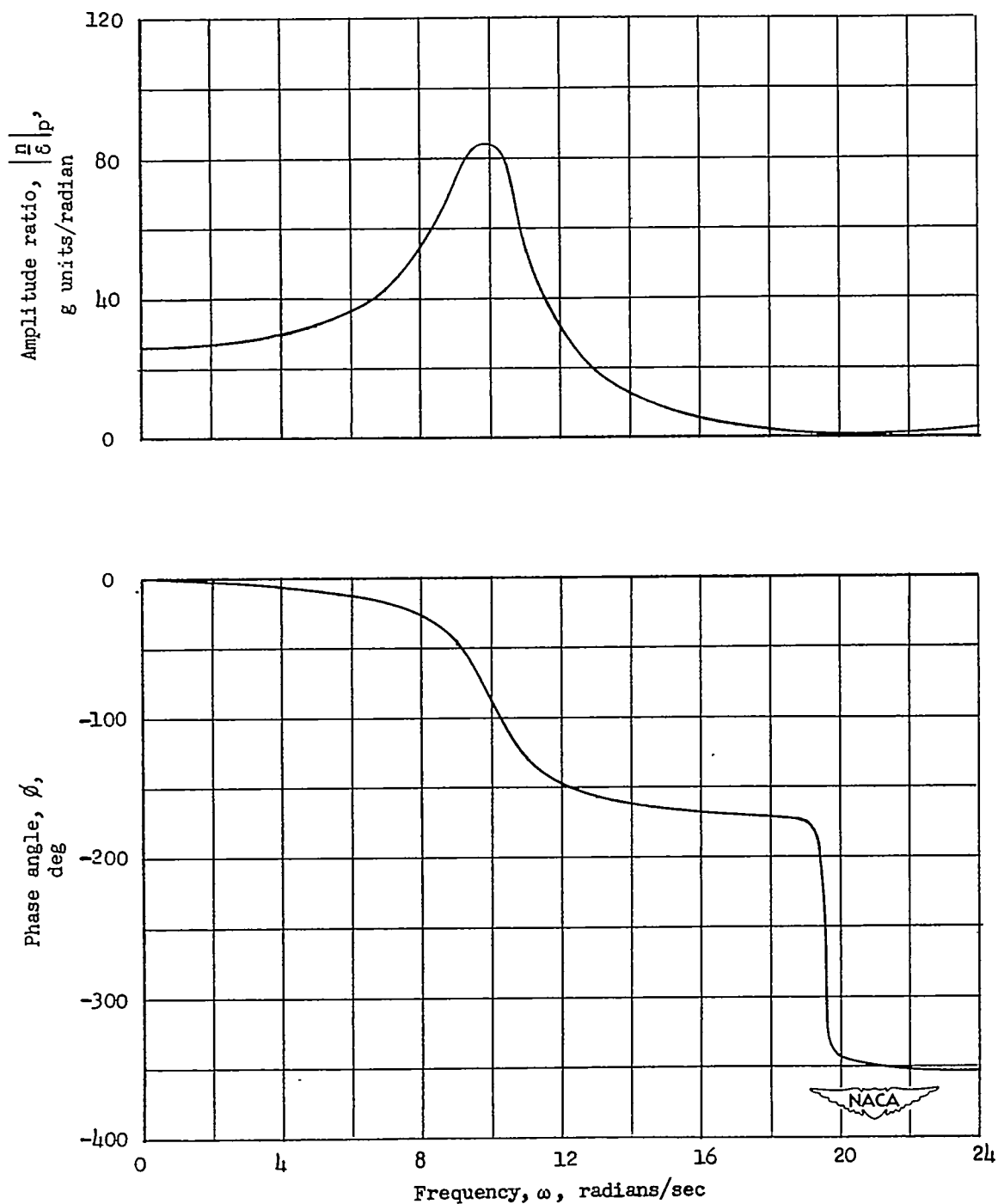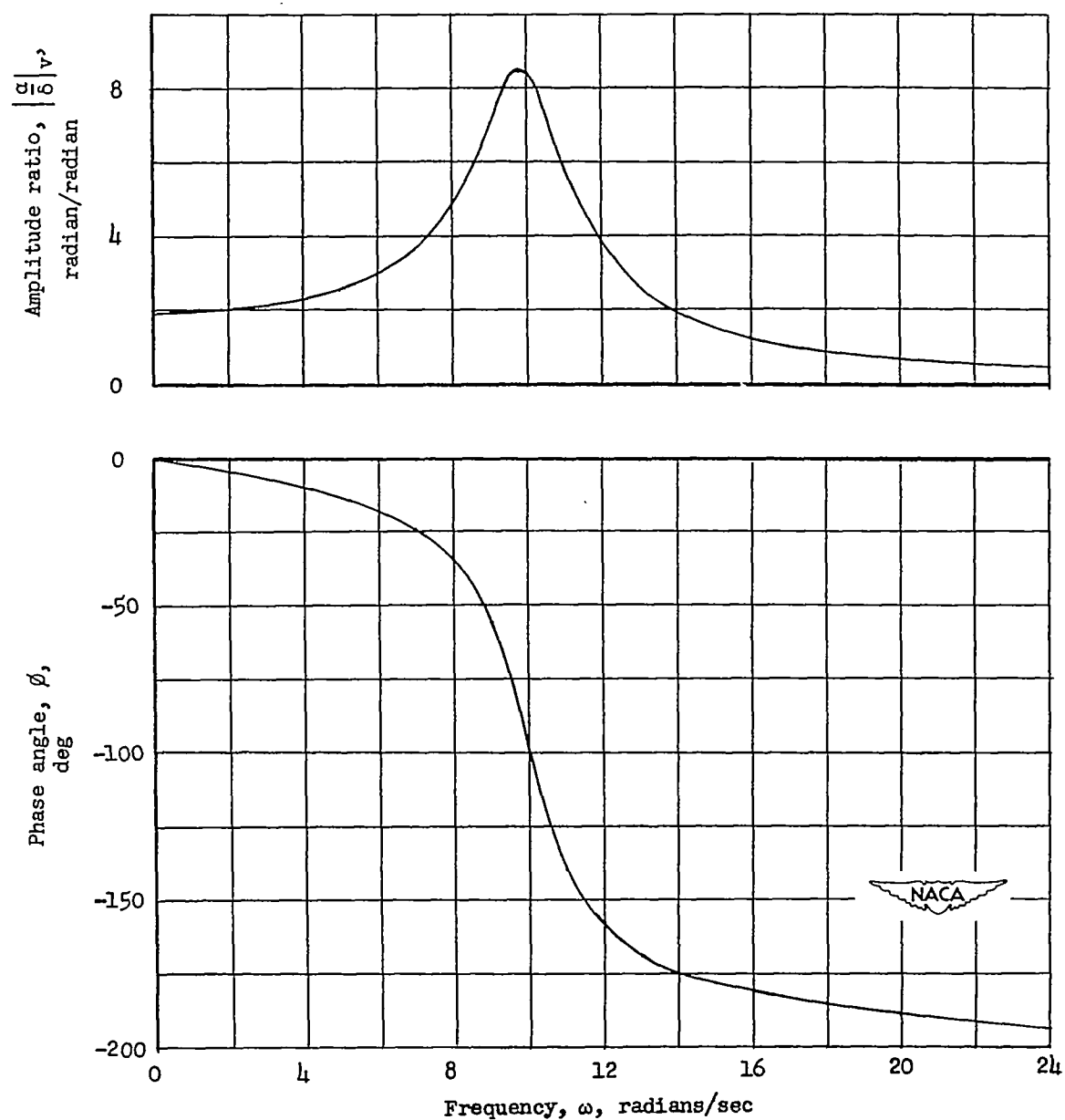Figure 1.- Sign convention. Arrows indicate positive direction.
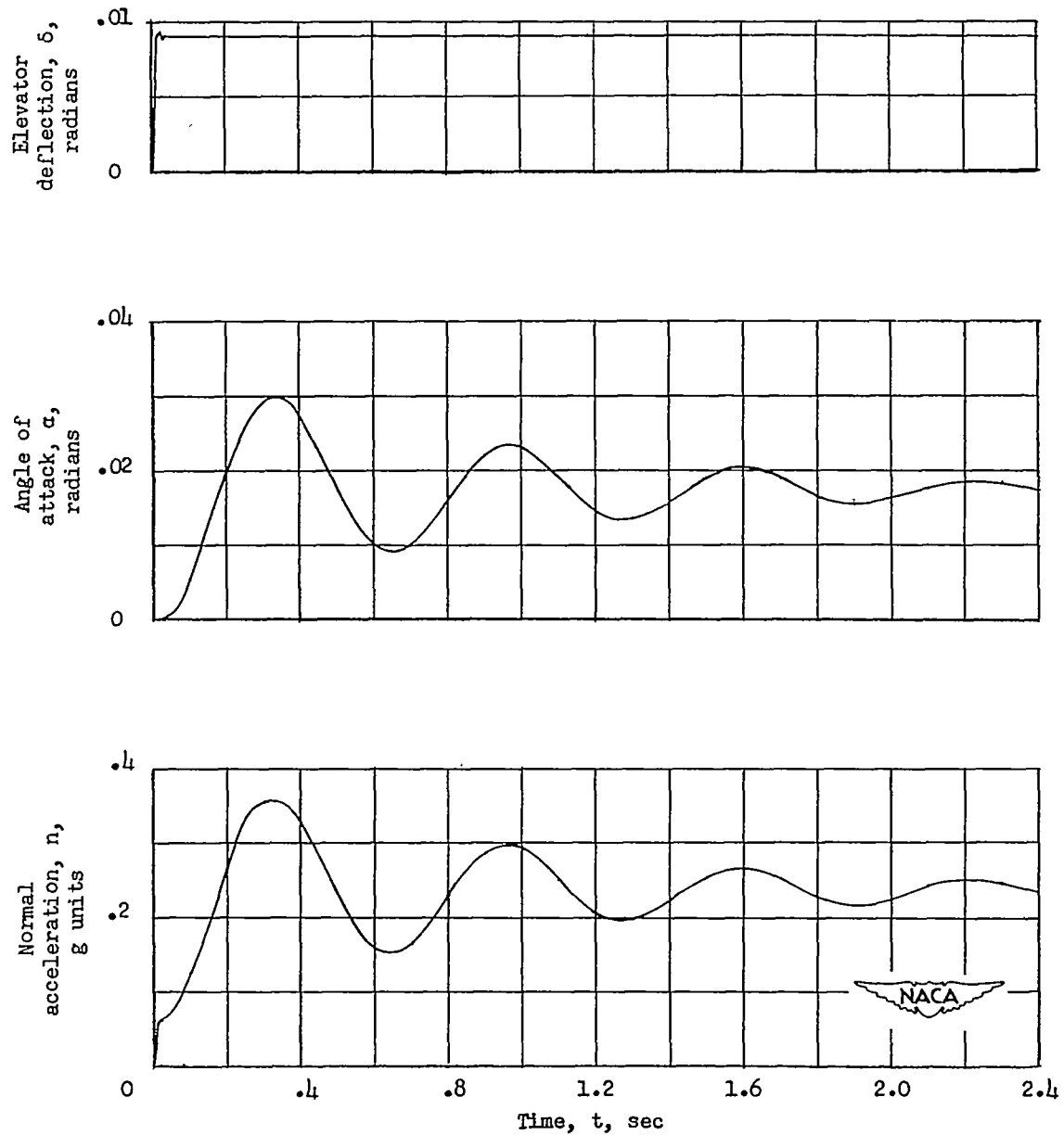
Figure 2.- Time history of elevator input and response in angle of attack
(measured 5.51 feet ahead of center of gravity) and normal acceleration
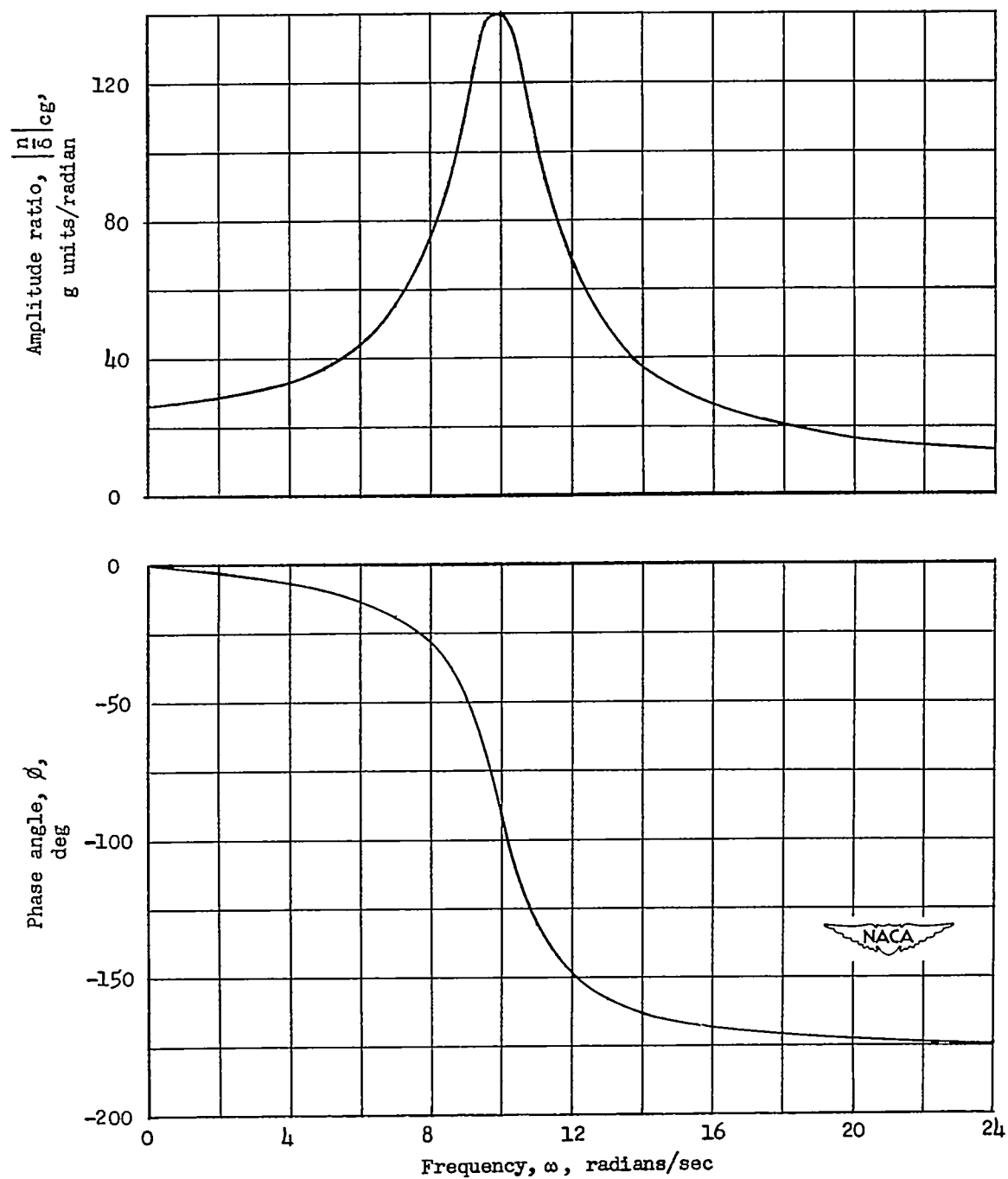(measured 2.165 feet ahead of center of gravity) of freely falling model.

(a) Normal-acceleration response 2.165 feet ahead of center of
gravity related to elevator deflection.

Figure 3.- Frequency response of freely falling model as obtained ahead
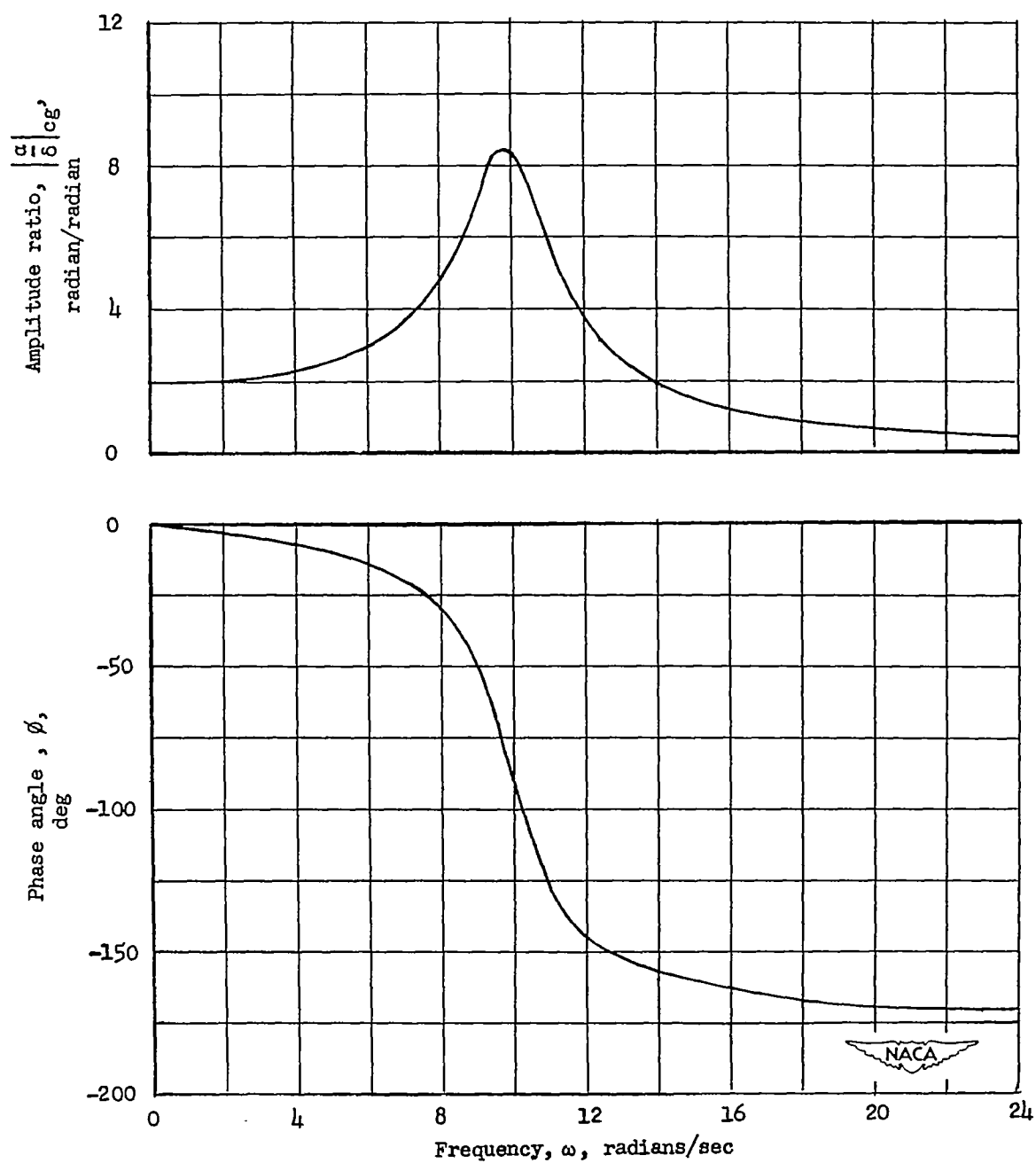of center of gravity.

(b) Angle-of-attack response 5.51 feet ahead of center of gravity
related to elevator deflection.
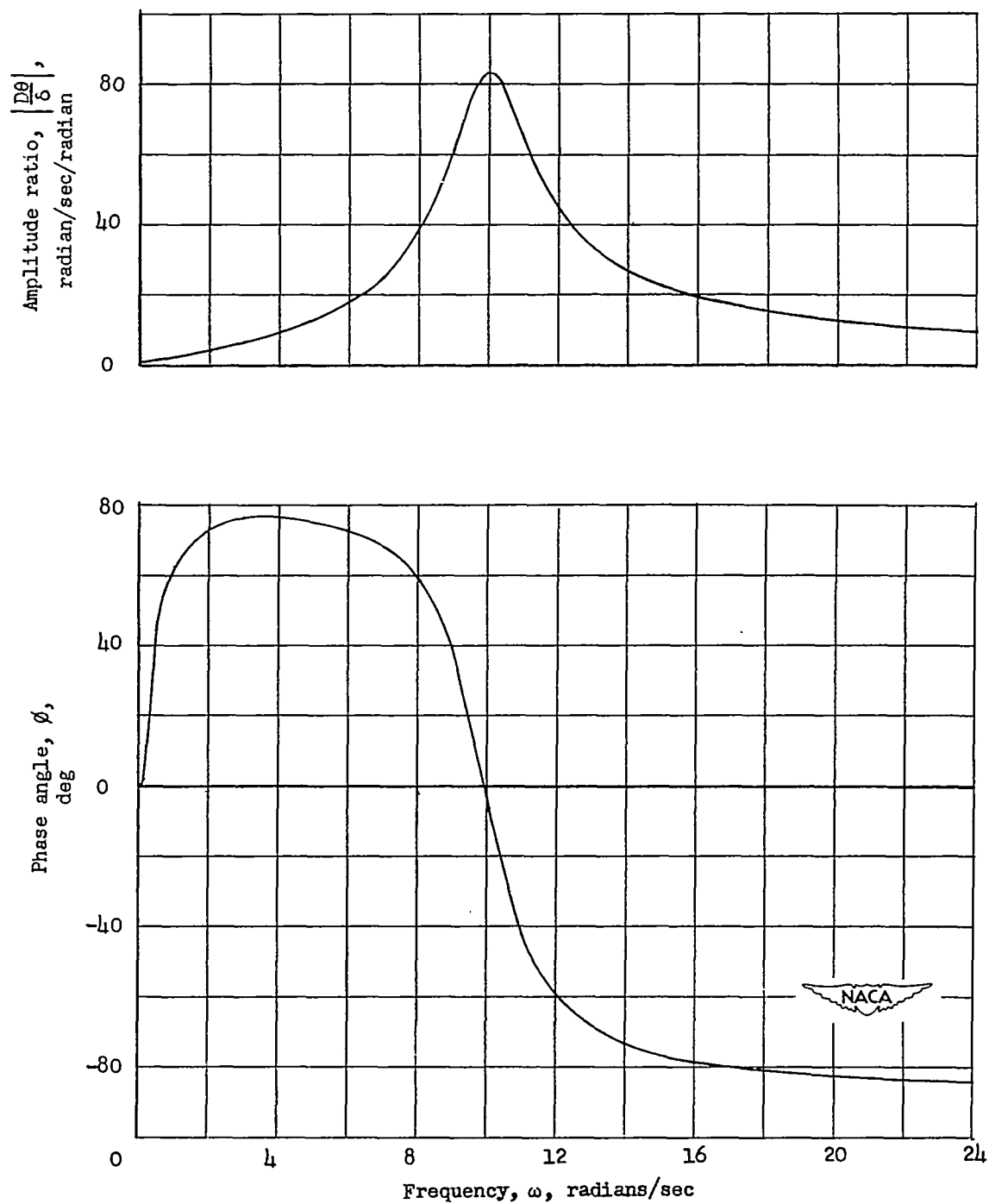
Figure 3.- Concluded.

(a) Normal-acceleration response at center of gravity related to elevator deflection.

Figure 4.- Frequency response of freely falling model at the center of gravity.

(b) Angle-of-attack response at center of gravity related to
elevator deflection.

Figure 4.- Continued.

(c) Pitching velocity related to elevator deflection.
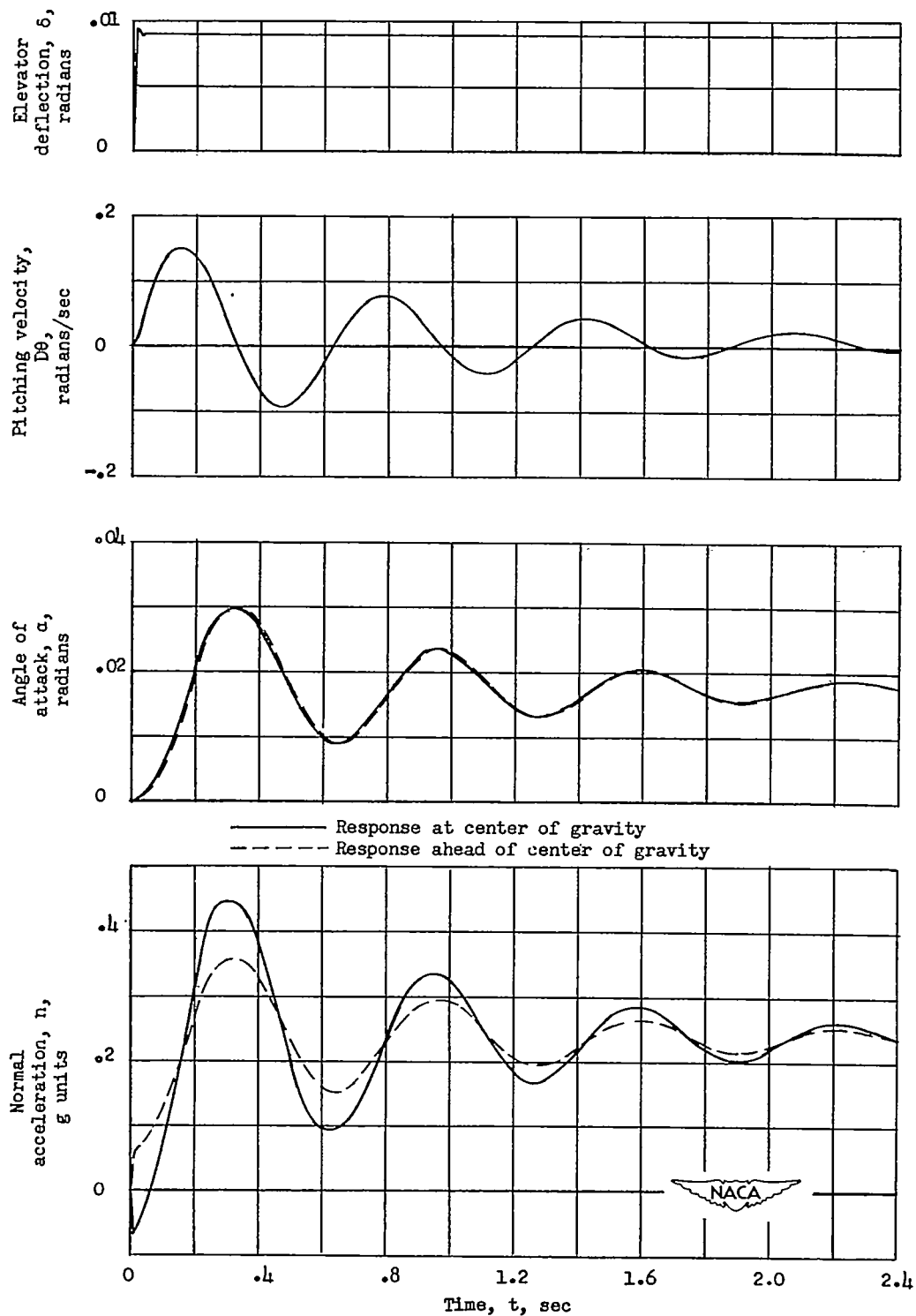
Figure 4.- Concluded.

Figure 5.- Time history of elevator input and response in angle of attack,
pitching velocity, and normal acceleration of the center of gravity
as compared with the response obtained at locations ahead of the center
of gravity of a freely falling model.